On Some New Classes of Ideal Convergent Triple Sequences of Fuzzy Numbers Associated with Multiplier Sequences

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Abstract: In this article, the notion of some new classes of multiplier ideal convergent fuzzy real-valued multiple sequence spaces having multiciplicity greater than two are introduced. The multiplier problem is characterized. Also we have made an effort to investigate some basic algebraic and topological properties of these introduced sequence spaces and investigate some inclusion results between these spaces.

Keywords: Triple sequences, fuzzy numbers, multiplier, ideal convergence, solid, symmetric, convergence free.

Introduction

The basic mathematical concept of a set was extended by the introduction of the fuzzy set theory. Fuzzy set theory is a powerful hand set for modeling, uncertainty and vagueness in various problems arising in the field of science and engineering such as cybernetics, artificial intelligence, expert system and fuzzy control, pattern recognition, operation research, decision making, image analysis, projectiles, probability theory, weather forecasting and so on. The concepts of fuzzy sets and fuzzy set operations were first introduced by Lofti A. Zadeh [33] in 1965 and after his pioneering work done on fuzzy set theory, a huge number of research papers have been appeared on fuzzy theory and its applications as well as fuzzy analogues of the classical theories. Several mathematicians have discussed various aspects of the theory and applications of fuzzy sets such as fuzzy topological spaces, similarity relations and fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming and so on. The theory of sequence of fuzzy numbers was first introduced by Matloka [12]. Matloka introduced bounded and convergent sequence of fuzzy numbers and showed that every convergent sequence of fuzzy numbers is bounded. Nanda [13] studied the sequences of fuzzy numbers and showed that the set of all convergent sequences of fuzzy numbers forms a complete metric space. The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical problems through the concept of density. The concept of ideal convergence as a generalization of statistical convergence was initially introduced by Kostyrko *et. al*. [9]. More investigations in this direction and more applications of ideals are found in Šalát *et al.* [19-20], Kumar and Kumar [11], Tripathy and Tripathy [32], Das *et. al*. [2], Tripathy and Sen [31], Tripathy and Hazarika [28], Sen and Roy [23], Khan and Khan [8], Raj and Gupta [16], Savas [23], Hazarika [7], Nath and Roy[14] and so on Agnew [1] introduced the summability theory of multiple sequences and proved certain theorems for double sequences. At the initial stage, the different types of notions of triple sequences were introduced and investigated by Sahiner *et. al.* [17] and Sahiner and Tripathy [18]. In 2012, Savas and Esi [22] have introduced statistical convergence of triple sequences on probabilistic normed space. Esi [4] introduced statistical convergence of triple sequences in topological groups. Recently more works on triple sequences are done by Kumar et. al. [10], Dutta et. al. [3], Tripathy and Goswami [27], Nath and Roy [15] and many others. Using the notion of associated multiplier sequences, the scope for the studies on sequence spaces was extended by several authors in several directions. In 1970, Goes and Goes [6] studied the notion of multiplier sequences and defined the differentiated sequence space *dE* and integrated sequence space $\int E$ for a given sequence space *E*, by using multiplier sequences $\left(k^{-1}\right)$ and (*k*) respectively. Later on Tripathy and Sen [30], Tripathy and Mahanta [29] used a general multiplier sequence $\left(\lambda_k\right)$ of non-zero scalars for their studies on sequence spaces. Sen and Roy [24-25] used a general multiplier sequence $\left(\lambda_{nk}\right)$ of non-zero scalars on double sequence spaces.

Preliminaries and Background

Throughout N, R and C denote the sets of natural and real numbers respectively.A fuzzy number on R is a function $X : R \to L(=[0,1])$ associating each real number $t \in R$ having grade of membership $X(t)$. We can express every real number r as a fuzzy number $\bar{r}$ as $\bar{r}(t) = \begin{cases} 1, & if \quad t = r \\ 0, & otherwise \end{cases}$

The α-level set of a fuzzy number X , $0 < \alpha \le 1$, is defined and denoted as

$$[X]^{\alpha} = \{t \in R : X(t) \ge \alpha\}.$$

A fuzzy number X is said to be convex if $X(t) \ge X(s) \wedge X(r) = \min(X(s), X(r))$,where $s < t < r$ and X is called normal if there exists $t_0 \in R$ such that $X(t_0) = 1$. If for each $\varepsilon > 0$, $X^{-1}[0, a+\varepsilon))$, for all $a \in L$ is open in the usual topology of R, then a fuzzy number X is said to be upper semi-continuous. The set of all upper semi continuous, normal, convex fuzzy number is denoted by $R(L)$, whose additive and multiplicative identities are denoted by $\bar{0}$ and $\bar{1}$ respectively.If D denotes the set of all closed bounded intervals $X = [X^L, X^R]$ on the real line R and if $d(X,Y) = \max(|X^L - X^R|, |Y^L - Y^R|)$, then (D,d) is a complete metric space. Also $\bar{d} : R(L) \times R(L) \to R$ defined by $\bar{d}(X,Y) = \sup_{0 \le \alpha \le 1} d([X]^{\alpha}, [Y]^{\alpha})$, for $X, Y \in R(L)$ is also a metric on $R(L)$. A non-void class $I \subseteq 2^X$ (power set of a non-empty set X) is said to be an ideal if I satisfies (i) $A, B \in I \Rightarrow A \cup B \in I$ and (ii) $A \in I$ *and* $B \subseteq A \Rightarrow B \in I$.

An ideal $I \subseteq 2^X$ is said to be non-trivial if $I \ne \emptyset$ and $X \notin I$.
A non-trivial ideal $I \subseteq 2^X$ is called admissible if I contains each finite subset of X. A non-trivial ideal I is called maximal if there does not exist any non-trivial ideal $J \ne I$ containing I as a subset. A non-empty family of sets $F \subseteq 2^X$ is said to be a filter on X if (i) $\emptyset \notin F$ (ii) $A, B \in F \Rightarrow A \cap B \in F \Rightarrow A \cap B \in F$ and (iii) $A \in F$ *and* $A \subseteq B \Rightarrow B \in F$.

For any ideal I, there is a filter $F(I)$ defined as $F(I) = \{K \subseteq N : N \setminus K \in I\}$. Throughout the article, the ideals of $2^{N \times N \times N}$ will be denoted by I_3 and ${}_3(w^F)$,${}_3(\ell_{\infty}^F)$, ${}_3(c^F)$, ${}_3(c_0{}^F)$ denote the spaces of all, bounded, convergent in Pringsheim's sense and null in Pringsheim's sense fuzzy real-valued triple sequences respectively.

A subset E of $N \times N \times N$ is said to have asymptotic density $\delta(E)$ if $\delta(E) = \lim_{p,q,r \to \infty} \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r} \chi_E(i,j,k)$ exists, where χ_E is the characteristic function of E.

A triple sequence is a function $x : N \times N \times N \to R(C)$.

A triple sequence $X = \langle X_{ijk} \rangle$ of fuzzy numbers is a triple infinite array of fuzzy numbers $X_{ijk} \in R(L)$ for all $i, j, k \in N$.

A triple sequence $X = \langle X_{ijk} \rangle$ of fuzzy numbers is said to be convergent in Pringsheim's sense to the fuzzy number X, if for every $\varepsilon > 0$, there exists $i_0 = i_0(\varepsilon), j_0 = j_0(\varepsilon), k_0 = k_0(\varepsilon) \in N$ such that $\bar{d}(X_{ijk}, X) < \varepsilon$, for all $i \ge i_0$, $j \ge j_0, k \ge k_0$.

A triple sequence $X = \langle X_{ijk} \rangle$ of fuzzy numbers is said to be I_3-convergent to the fuzzy number X_0, if for all $\varepsilon > 0$, the set $\{(i,j,k) \in N \times N \times N : \bar{d}(X_{ijk}, X_0) \ge \varepsilon\} \in I_3$ and we write $I_3 - \lim X_{ijk} = X_0$. A triple sequence space E^F of fuzzy numbers is said to be normal or solid if $\langle Y_{ijk} \rangle \in E^F$ whenever $\langle X_{ijk} \rangle \in E^F$ and $\bar{d}(Y_{ijk}, \bar{0}) \le \bar{d}(X_{ijk}, \bar{0})$ for all $i, j, k \in N$. A K-step space of a triple sequence space E^F of fuzzy numbers is a sequence space $\lambda_K^{E^F} = \{(X_{i_n j_n k_n}) \in_3 (w^F) : (X_{ijk}) \in E^F\}$. A canonical pre-image of a sequence $(X_{i_n j_n k_n}) \in E^F$ is a sequence $\langle Y_{ijk} \rangle \in_3 (w^F)$ defined as :

$$Y_{ijk} = \begin{cases} X_{ijk}, & if \quad (i,j,k) \in K \\ \bar{0}, & otherwise \end{cases}.$$

A canonical pre-image of a step space $\lambda_K^{E^F}$ is a set of canonical pre- images of all elements in $\lambda_K^{E^F}$.

A triple sequence space E^F of fuzzy numbers is said to be monotone if E^F contains the canonical pre-image of all its step spaces.

A triple sequence space E^F of fuzzy numbers is said to be symmetric if $\langle X_{\pi(ijk)}\rangle \in E^F$, whenever $\langle X_{ijk}\rangle \in E^F$ where π is a permutation on $N \times N \times N$.
A triple sequence space E^F of fuzzy numbers is said to be sequence algebra if $\langle X_{ijk} \otimes Y_{ijk}\rangle \in E^F$, whenever $\langle X_{ijk}\rangle, \langle Y_{ijk}\rangle \in E^F$.
A triple sequence space E^F of fuzzy numbers is said to be convergence free if $\langle Y_{ijk}\rangle \in E^F$ whenever $\langle X_{ijk}\rangle \in E^F$ and $X_{nlk} = \bar{0}$ implies $Y_{ijk} = \bar{0}$.
If $\Lambda = \langle \lambda_{ijk}\rangle$ is a triple sequence of non-zero scalars, then for a sequence space E^F of fuzzy numbers, the multiplier sequence space $E^F(\Lambda)$ is defined as

$$E^F(\Lambda) = \{\langle X_{ijk}\rangle : \langle \lambda_{ijk} X_{ijk}\rangle \in E^F\}.$$

A multiplier from a sequence space D^F into another sequence space E^F is a real sequence $u = \langle u_{ijk}\rangle$ such that $uX = \langle u_{ijk} X_{ijk}\rangle \in E^F$, whenever $X = \langle X_{ijk}\rangle \in D^F$. We denote the linear space of all such multipliers and bounded multipliers by $m(D^F, E^F)$ and $M(D^F, E^F)$ respectively. In fact $M(D^F, E^F) = {}_3(\ell^F_\infty) \cap m(D^F, E^F)$.
Let $\langle X_{ijk}\rangle$ and $\langle Y_{ijk}\rangle$ be two triple sequences of fuzzy numbers. Then $X_{ijk} = Y_{ijk}$ for almost all *i, j and k* relative to I_3 (in short *a.a. i, j & k* $r. I_3$) if $\{(i,j,k) \in N \times N \times N : X_{ijk} \neq Y_{ijk}\} \in I_3$.
The following well-known inequality will be used throughout the paper.
If $p = \langle p_{ijk}\rangle$ is a triple sequence of bounded strictly positive numbers and $H = \sup_{i,j,k} p_{ijk} < \infty$, then for two triple sequences $\langle a_{ijk}\rangle$ and $\langle b_{ijk}\rangle$, $|a_{ijk} + b_{ijk}|^{p_{nkl}} \le D(|a_{ijk}|^{p_{ijk}} + |b_{ijk}|^{p_{ijk}})$, where $D = \max(1, 2^{H-1})$.
To prove some results in the paper, we will use the following Lemmas.
Lemma 2.1 Every solid sequence space is monotone.
Lemma 2.2 If a sequence space E^F of fuzzy numbers is bounded and normal, then $\langle \lambda_{ijk}\rangle \in M(E^F, E^F)$ if and only if $\langle \lambda_{ijk}\rangle \in {}_3\ell_\infty$.
If $\Lambda = \langle \lambda_{ijk}\rangle$ is a multiplier sequence, $p = \langle p_{ijk}\rangle$ is a triple sequence of bounded strictly positive numbers and $X = \langle X_{ijk}\rangle$ is a sequence of fuzzy numbers, we introduce the following sequence spaces:

$${}_3(c^F)(\Lambda, p) = \{X = \langle X_{ijk}\rangle \in {}_3(w^F) : \lim [\bar{d}(\lambda_{ijk} X_{ijk}, X_0)]^{p_{ijk}} = 0, \text{ for some } X_0 \in R(L)\},$$

$${}_3(c^{I(F)})(\Lambda, p) = \{X = \langle X_{ijk}\rangle \in {}_3(w^F) : I_3\text{-}\lim [\bar{d}(\lambda_{ijk} X_{ijk}, X_0)]^{p_{ijk}} = 0, \text{ for some } X_0 \in R(L)\},$$

$${}_3(c_0^{I(F)})(\Lambda, p) = \{X = \langle X_{ijk}\rangle \in {}_3(w^F) : I_3\text{-}\lim \bar{d}(\lambda_{ijk} X_{ijk}, \bar{0})]^{p_{ijk}} = 0\},$$

$${}_3(\ell_\infty^{(F)})(\Lambda, p) = \left\{X = \langle X_{ijk}\rangle \in {}_3(w^F) : \sup_{i,j,k} \lim \bar{d}(\lambda_{ijk} X_{ijk}, \bar{0})]^{p_{ijk}} < \infty\right\}.$$

$${}_3(m^{I(F)})(\Lambda, p) = {}_3(c^{I(F)})(\Lambda, p) \cap {}_3(\ell_\infty^{(F)})(\Lambda, p)$$

and $${}_3(m_0^{I(F)})(\Lambda, p) = {}_3(c_0^{I(F)})(\Lambda, p) \cap {}_3(\ell_\infty^{(F)})(\Lambda, p).$$

Main Results

Theorem 3.1 The sequence spaces ${}_3(m^{I(F)})(\Lambda, p)$ and ${}_3(m_0^{I(F)})(\Lambda, p)$ are are closed under addition and multiplication operations.

Proof. We shall prove the result for the space ${}_3(m_0^{I(F)})(\Lambda, p)$ and the other can be proved in a similar way.
Let $\langle X_{ijk}\rangle, \langle Y_{ijk}\rangle \in {}_3(m_0^{I(F)})(\Lambda, p)$. Then the sets

$$A=\left\{(i,j,k)\in N\times N\times N:[\bar{d}(\lambda_{ijk}X_{ijk},\bar{0})]^{p_{ijk}}\geq\frac{\varepsilon}{2}\right\}\in I_3$$ and

$$B=\left\{(i,j,k)\in N\times N\times N:[\bar{d}(\lambda_{ijk}Y_{ijk},\bar{0})]^{p_{ijk}}\geq\frac{\varepsilon}{2}\right\}\in I_3.$$

Let α,β be scalars. Now, we have

$$[\bar{d}(\lambda_{ijk}(\alpha X_{ijk}+\beta Y_{ijk}),\bar{0}]^{p_{ijk}}\leq D[\bar{d}(\lambda_{ijk}X_{ijk},\bar{0})]^{p_{ijk}}+D[\bar{d}(\lambda_{ijk}Y_{ijk},\bar{0})]^{p_{ijk}}\ ....(1)$$

where $D=\max(1,2^{H-1})$, $H=\sup\limits_{i,j,k}p_{ijk}<\infty.$ From (1), we obtained

$$\left\{(i,j,k)\in N\times N\times N:[\bar{d}(\lambda_{ijk}(\alpha X_{ijk}+\beta Y_{ijk}),\bar{0})]^{p_{ijk}}\geq\frac{\varepsilon}{2}\right\}\subseteq$$

$$\left\{(i,j,k)\in N\times N\times N:D[\bar{d}(\lambda_{ijk}(X_{ijk},\bar{0})]^{p_{ijk}}\geq\frac{\varepsilon}{2}\right\}\cup\left\{(i,j,k)\in N\times N\times N:D[\bar{d}(\lambda_{ijk}(Y_{ijk},\bar{0})]^{p_{ijk}}\geq\frac{\varepsilon}{2}\right\}\in I_3.\ \therefore(\alpha X_{ijk}+\beta Y_{ijk})\in{}_3(m^{I(F)})(\Lambda,p).$$ ■

Theorem 3.2 Let $\sup\limits_{i,j,k}p_{ijk}<\infty.$ Then the following statements are equivalent:

(i) $\langle X_{ijk}\rangle\in{}_3(c^{I(F)})(\Lambda,p).$

(ii) $\exists$ a triple sequence $\langle Y_{ijk}\rangle\in{}_3(c^F)(\Lambda,p)$ such that $X_{ijk}=Y_{ijk}$ for a.a. i, j & k r I_3.

(iii) $\exists$ a subset $M=\{(i_n,j_m,k_l)\in N\times N\times N:n,m,l\in N\}$ of $N\times N\times N$ such that $M\in F(I_3)$ and $(X_{i_nj_mk_l})\in{}_3(c^F)(\Lambda,p).$

Proof. (i) $\Rightarrow$ (ii). Let $\langle X_{ijk}\rangle\in{}_3(c^{I(F)})(\Lambda,p).$ Then $\exists$ $X_0\in R(L)$ such that

$$I_3-\lim[\bar{d}(\lambda_{ijk}X_{ijk},X_0)]^{p_{ijk}}=0$$

So for any $\varepsilon>0$, $\{(i,j,k)\in N\times N\times N:[\bar{d}(\lambda_{ijk}X_{ijk},X_0)]^{p_{ijk}}\geq\varepsilon\}\in I_3.$

Let (S_m), (T_m) and (U_m) be three increasing sequences of natural numbers such that if $p>S_m$, $q>T_m$ and $r>U_m$, then

$$\left\{(i,j,k)\in N\times N\times N:i\leq p,\ j\leq q,k\leq r\ \text{and}\ [\bar{d}(\lambda_{ijk}X_{ijk},X_0)]^{p_{ijk}}\geq\frac{1}{m}\right\}\in I_3.$$

We define the sequence $\langle Y_{ijk}\rangle$ by:

$$Y_{ijk}=X_{ijk},\ \text{if}\ i\leq S_1\ \text{or}\ j\leq T_1\ \text{or}\ k\leq U_1.$$

Also for all (i,j,k) with $S_m<i\leq S_{m+1}$ or $T_m<j\leq T_{m+1}$ or $U_m<k\leq U_{m+1}$,

let $Y_{ijk}=X_{ijk}$, if $[\bar{d}((\lambda_{ijk}X_{ijk},X_0)]^{p_{ijk}}<\frac{1}{m}$, otherwise $Y_{ijk}=\lambda_{ijk}{}^{-1}X_0$.

Let $\varepsilon>0$ and m be chosen such that $\varepsilon>\frac{1}{m}$. For $i>S_m$, $j>T_m$ and $k>U_m$, $[\bar{d}((\lambda_{ijk}X_{ijk},X_0)]^{p_{ijk}}<\varepsilon.$

Hence $\langle Y_{ijk}\rangle\in{}_3(c^F)(\Lambda,p).$

Next let $S_m<i\leq S_{m+1}$, $T_m<j\leq T_{m+1}$ and $U_m<k\leq U_{m+1}$, then

$$A=\{(i,j,k)\in N\times N\times N:X_{ijk}\neq Y_{ijk}\}\subseteq\left\{(i,j,k)\in N\times N\times N:[\bar{d}((\lambda_{ijk}X_{ijk},X_0)]^{p_{ijk}}\geq\frac{1}{m}\right\}\in I_3.$$ Hence $A\in I_3$ and so $X_{ijk}=Y_{ijk}$ for a.a. i, j & k r. I_3.

(ii) $\Rightarrow$ (iii). Let there exists a triple sequence $\langle Y_{ijk}\rangle\in{}_3(c^F)(\Lambda,p)$ such that $X_{ijk}=Y_{ijk}$ for a.a i, j. & k r. I_3.

If $M = \left\{ (i, j, k) \in N \times N \times N : X_{ijk} = Y_{ijk} \right\}$, then $M \in F(I_3)$. On neglecting the rows and columns those contain finite number of elements, M can be enumerated as $M = \{(i_n, j_m, k_l) \in N \times N \times N : n, m, l \in N\}$. Then $(X_{i_n j_m k_j}) \in {}_3(c^F)(\Lambda, p)$.

(iii) $\Rightarrow$ (i). From (iii), the result (i) follows immediately. ■

Theorem 3.3 The sequence spaces ${}_3(m^{I(F)})(\Lambda, p)$ and ${}_3(m_0^{I(F)})(\Lambda, p)$ are complete metric spaces with respect to the metric ρ defined as $\rho(X,Y) = \sup_{i,j,k} [\bar{d}(\lambda_{ijk} X_{ijk}, \lambda_{ijk} Y_{ijk})]^{\frac{p_{ijk}}{M}}$, where $M = \max(1, H)$, $H = \sup_{i,j,k} p_{ijk} < \infty$. Proof. Consider the space ${}_3(m^{I(F)})(\Lambda, p)$.

Let $\langle X^{(n)} \rangle$ be a Cauchy sequence in ${}_3(m^{I(F)})(\Lambda, p) \subset {}_3(\ell_\infty^{(F)})(\Lambda, p)$, where $X^{(n)} = \langle X_{ijk}^{(n)} \rangle$. Since ${}_3(\ell_\infty^{(F)})(\Lambda, p)$ is complete, so $\exists$ $X \in {}_3(\ell_\infty^{(F)})(\Lambda, p)$ such that $\lim_{n\to\infty} X^{(n)} = X$, where $X = \langle X_{ijk} \rangle$.

Since $\langle X^{(n)} \rangle$ is Cauchy, so for a given $0 < \varepsilon < 1$, $\exists$ $n_0 \in N$ such that

$$\rho\left(X^{(n)}, X^{(m)}\right) < \frac{\varepsilon}{3}, \text{ for all } n, m \ge n_0.$$

$$\Rightarrow \left[\bar{d}(\lambda_{ijk} X_{ijk}^{(n)}, \lambda_{ijk} X_{ijk}^{(m)})\right]^{\frac{p_{nlk}}{M}} < \frac{\varepsilon}{3}, \text{ for all } n, m \ge n_0.$$

$$\Rightarrow \bar{d}(\lambda_{ijk} X_{ijk}^{(n)}, \lambda_{ijk} X_{ijk}^{(m)}) < \left(\frac{\varepsilon}{3}\right)^{\frac{M}{p_{nlk}}}, \text{ for all } n, m \ge n_0.$$

Again since $X^{(n)}, X^{(m)} \in {}_3(m^{I(F)})(\Lambda, p)$, so $\exists$ fuzzy numbers Y_n and Y_m such that

$$A = \left\{ (i, j, k) \in N \times N \times N : \left[\bar{d}(\lambda_{ijk} X_{ijk}^{(n)}, Y_i)\right]^{p_{ijk}} < \left(\frac{\varepsilon}{3}\right)^M \right\} \in F(I_3)$$

and

$$B = \left\{ (i, j, k) \in N \times N \times N : \left[\bar{d}(\lambda_{ijk} X_{ijk}^{(m)}, Y_j)\right]^{p_{nlk}} < \left(\frac{\varepsilon}{3}\right)^M \right\} \in F(I_3).$$

Then $A \cap B \in F(I_3)$. Let $(i, j, k) \in A \cap B$.

Now $\bar{d}(Y_n, Y_m) \le \bar{d}(Y_n, \lambda_{ijk} X_{ijk}^{(n)}) + \bar{d}(\lambda_{ijk} X_{ijk}^{(n)}, \lambda_{ijk} X_{ijk}^{(m)}) + \bar{d}(\lambda_{ijk} X_{ijk}^{(m)}, Y_m)$

$$< \varepsilon, \text{ for all } n, m \ge n_0.$$

Hence $\langle Y_n \rangle$ is a Cauchy sequence fuzzy numbers. So there exists a fuzzy number Y such that $\lim_{n\to\infty} Y_n = Y$. Let $\eta > 0$ be given. Since $X^{(n)} \to X$, so $\exists$ $t \in N$ such that

$$\rho\left(X^{(t)}, X\right) < \left(\frac{\eta}{3}\right)^{\frac{1}{M}} \quad \text{.........} \quad (1)$$

Now together with (1), a number t is chosen such that $\left[\bar{d}(Y_t, Y)\right]^{p_{ijk}} < \frac{\eta}{3}$. Since $\langle X_{ijk}^{(t)} \rangle$ is I-convergent to Y_t, so

$$C = \left\{ (i, j, k) \in N \times N \times N : \left[\bar{d}(\lambda_{ijk} X_{ijk}^{(t)}, Y_i)\right]^{p_{ijk}} < \frac{\eta}{3} \right\} \in F(I_3).$$

So for each $(i, j, k) \in C$,

$\left[\bar{d}(\lambda_{ijk}X_{ijk}, Y)\right]^{p_{ijk}} \leq D^2\left[\bar{d}(\lambda_{ijk}X_{ijk}, \lambda_{ijk}X_{ijk}^{(t)})\right]^{p_{ijk}} + D^2\left[\bar{d}(\lambda_{ijk}X_{ijk}^{(t)}, Y_t)\right]^{p_{ijk}} + D\left[\bar{d}(Y_t, Y)\right]^{p_{ijk}} \leq D^2\left(\frac{\eta}{3}\right) + D^2\left(\frac{\eta}{3}\right) + D\left(\frac{\eta}{3}\right) = \eta'$ (say), where $D = \max(1, 2^{H-1})$, $H = \sup_{i,j,k} p_{ijk} < \infty$.

Hence $\langle X_{ijk} \rangle$ is I-convergent to Y which implies ${}_3(m^{I(F)})(\Lambda, p)$ is complete. Using similar technique, we can prove the result for the other space. ■

Theorem 3.4 The sequence space ${}_3(m_0^{I(F)})(\Lambda, p)$ is normal and monotone.

Proof. Let $\langle X_{ijk} \rangle \in {}_3(m_0^{I(F)})(\Lambda, p)$ and $\langle Y_{ijk} \rangle$ be such that $\bar{d}(Y_{ijk}, \bar{0}) \leq \bar{d}(X_{ijk}, \bar{0})$, for all $i, j, k \in N$. Let $\varepsilon > 0$ be given. Then from the following inclusion relation $\{(i,j,k) \in N \times N \times N : [\bar{d}(\lambda_{ijk}X_{ijk}, \bar{0})]^{p_{ijk}} \geq \varepsilon\} \supseteq \{(i,j,k) \in N \times N \times N : [\bar{d}(\lambda_{ijk}Y_{ijk}, \bar{0})]^{p_{ijk}} \geq \varepsilon\}$. it follows that ${}_3(m_0^{I(F)})(\Lambda, p)$ is normal. By Lemma 2.1, the space is monotone. ■ Proposition 3.5 The sequence space ${}_3(m^{I(F)})(\Lambda, p)$ is neither solid nor monotone.

Proof. To prove the result, we give a counter example.

Example 3.1 Let $I_3 = I_3(\rho)$, $A \in I_3$, $p_{ijk} = 1$ and $\lambda_{ijk} = \dfrac{1}{(i+j+k)}$, for all $i, j, k \in N$.

We define the sequence $\langle X_{ijk} \rangle$ as:

For all $(i, j, k) \notin A$,

$$X_{ijk}(t) = \begin{cases} 1 + (i+j+k)(t-3), & for\ \ 3 - \dfrac{1}{(i+j+k)} \leq t \leq 3 \\ 1 - (i+j+k)(t-3), & for\ \ 3 < t \leq 3 + \dfrac{1}{(i+j+k)} \\ 0, & otherwise \end{cases}$$

Otherwise $X_{ijk} = \bar{1}$.

Then $\langle X_{ijk} \rangle \in_3 (m^{I(F)})(\Lambda, p)$.

Let $K = \{(i, j, k) : i + j + k = 3q : q \in N\}$

The sequence $\langle Y_{ijk} \rangle$ is defined by:

$$Y_{ijk} = \begin{cases} X_{ijk}, & if\ \ (i,j,k) \in K \\ \bar{0}, & otherwise \end{cases}$$

Then $\langle Y_{ijk} \rangle$ belongs to the canonical pre-image of K step space of ${}_3(m^{I(F)})(\Lambda, p)$. But $\langle Y_{ijk} \rangle \notin_3 (m^{I(F)})(\Lambda, p)$, which implies that ${}_3(m^{I(F)})(\Lambda, p)$ is not monotone. Therefore ${}_3(m^{I(F)})(\Lambda, p)$ is not normal. ■

Proposition 3.6 The sequence spaces ${}_3(m^{I(F)})(\Lambda, p)$ and ${}_3(m_0^{I(F)})(\Lambda, p)$ are not symmetric.

Proof. The result follows from the following example.

Example 3.2. Let $I_3 = I_3(\rho)$, $p_{ijk} = \begin{cases} 1, & for\ \ i\ \ even \text{ and } all\ \ j, k \in N \\ 2, & otherwise \end{cases}$

Let the sequence $\langle X_{ijk} \rangle$ be defined as:

For $i = n^3$, $n \in N$ and for all $j, k \in N$,

$$X_{ijk}(t) = \begin{cases} 1 + \dfrac{t}{3\sqrt[3]{i} - 2}, & for\ \ 2 - 3\sqrt[3]{i} \leq t \leq 0 \\ 1 - \dfrac{t}{3\sqrt[3]{i} - 2}, & for\ \ 0 < t \leq 3\sqrt[3]{i} - 2 \\ 0, & otherwise \end{cases}$$

Otherwise $X_{ijk} = \bar{0}.$ Then taking $\lambda_{ijk} = \frac{1}{i},$ for all $i, j, k \in N,$ we have $\langle X_{ijk} \rangle \in {}_3(m^{I(F)})(\Lambda, p), {}_3(m_0^{I(F)})(\Lambda, p).$

Define the rearrangement $\langle Y_{ijk} \rangle$ of $\langle X_{ijk} \rangle$ as:

For j, k odd and for all $i \in N,$

$$Y_{ijk}(t) = \begin{cases} 1 + \frac{t}{3i-2}, & for \quad 2-3i \le t \le 0 \\ 1 - \frac{t}{3i-2}, & for \quad 0 < t \le 3i-2 \\ 0, & otherwise \end{cases}$$

Otherwise $Y_{ijk} = \bar{0}.$

Then $\langle Y_{ijk} \rangle \notin {}_3(m^{I(F)})(\Lambda, p), {}_3(m_0^{I(F)})(\Lambda, p).$ ■

Proposition 3.7 The sequence spaces ${}_3(m^{I(F)})(\Lambda, p)$ and ${}_3(m_0^{I(F)})(\Lambda, p)$ are not sequence algebras.

Proof. The result follows from the following example.

Example 3.3 Let $A \in I_3,$ $p_{ijk} = \begin{cases} \frac{1}{3}, & ij \ (i, j, k) \in A \\ 1, & otherwise \end{cases}$

Define the sequences $\langle X_{ijk} \rangle$ and $\langle Y_{ijk} \rangle$ by:

For all $(i, j, k) \notin A,$

$$X_{ijk}(t) = \begin{cases} 1 + \frac{t}{(i+j+k)^3}, & for \ -(i+j+k)^3 \le t \le 0 \\ 1 - \frac{t}{(i+j+k)^3}, & for \ 0 < t \le (i+j+k)^3 \\ 0, & otherwise \end{cases}$$

Otherwise $X_{ijk} = \bar{0}.$

For all $(i, j, k) \notin A,$

$$Y_{ijk}(t) = \begin{cases} 1 + \frac{t-1}{(i+j+k)^3}, & for \ 1-(i+j+k)^3 \le t \le 1 \\ 1 - \frac{t-1}{(i+j+k)^3}, & for \ 1 < t \le 1+(i+j+k)^3 \\ 0, & otherwise \end{cases}$$

Otherwise $Y_{ijk} = \bar{0}.$

Then taking $\lambda_{ijk} = \frac{1}{(i+j+k)^4},$ for all $i, j, k \in N,$

$\langle X_{ijk} \rangle, \langle Y_{ijk} \rangle \in {}_3(m^{I(F)})(\Lambda, p), {}_3(m_0^{I(F)})(\Lambda, p).$ But $(X_{nk} \otimes Y_{nk}) \notin {}_3(m^{I(F)})(\Lambda, p), {}_3(m_0^{I(F)})(\Lambda, p).$ ■

Proposition 3.8 The sequence spaces ${}_3(m^{I(F)})(\Lambda, p)$ and ${}_3(m_0^{I(F)})(\Lambda, p)$ are not convergence free.

Proof. From the following example, the result follows.

Example 3.4 Let $A \in I_3,$ $p_{ijk} = \frac{1}{3},$ for all $i, j, k \in N,$

Consider the sequences $\langle X_{ijk} \rangle$ defined by:

For all $(i, j, k) \notin A,$

$$X_{ijk}(t)=\begin{cases}1+3(i+j+k)t, & for\ \frac{1}{3(i+j+k)}\le t\le 0\\ 1-3(i+j+k)t, & for\ 0<t\le\frac{1}{3(i+j+k)}\\ 0, & otherwise\end{cases}$$

Otherwise $X_{ijk}=\bar{0}.$

Then taking $\lambda_{ijk}=\frac{1}{(i+j+k)},$ for all $i,j,k\in N,$ it follows that $\langle X_{ijk}\rangle\in {}_3(m^{I(F)})(\Lambda,p),\ {}_3(m_0^{I(F)})(\Lambda,p).$

Next consider the sequence $\langle Y_{ijk}\rangle$ defined as:

For all $(i,j,k)\notin A,$

$$Y_{ijk}(t)=\begin{cases}1+\frac{3t}{i+j+k}, & for\ -\frac{i+j+k}{3}\le t\le 0\\ 1-\frac{3t}{i+j+k}, & for\ 0<t\le\frac{i+j+k}{3}\\ 0, & otherwise\end{cases}$$

Otherwise $Y_{ijk}=\bar{0}.$

But $\langle Y_{ijk}\rangle\notin {}_3(m^{I(F)})(\Lambda,p),\ {}_3(m_0^{I(F)})(\Lambda,p).$

Hence the two sequence spaces are not convergence free. ■

Theorem 4.9 $(\lambda_{ijk})\in M\left({}_3(c_0{}^{I(F)})(p),\ {}_3(c_0{}^{I(F)})(p)\right)$ if and only if $(\lambda_{ijk})\in {}_3(\ell_\infty^I)(p).$

Proof. Let $(\lambda_{ijk})\in {}_3(\ell_\infty^I)(p)$ and $\langle X_{ijk}\rangle\in {}_3(c_0{}^{I(F)})(p).$

Then $\exists\ J>0$ such that

$$P=\{(i,j,k)\in N\times N\times N:\left|\lambda_{ijk}\right|^{p_{ijk}}\le J\}\in F(I_3)$$ and

$$Q=\left\{(i,j,k)\in N\times N\times N:[\bar{d}(X_{ijk},\bar{0})]^{p_{ijk}}<\frac{\varepsilon}{J}\right\}\in F(I_3).$$

Then $P\cap Q=\{(i,j,k)\in N\times N\times N:[\bar{d}(\lambda_{ijk}X_{ijk},\bar{0})]^{p_{ijk}}<\varepsilon\}\in F(I_3).$

Hence $\left(\lambda_{ijk}X_{ijk}\right)\in {}_3(c_0^{I(F)})(p)$ and so $(\lambda_{ijk})\in M\left({}_3(c_0{}^{I(F)})(p),\ {}_3(c_0{}^{I(F)})(p)\right).$ As the converse part is simple, so we omit its prove. ■

Theorem 4.10 If the sequence space ${}_3(c^{I(F)})(p)$ is not normal, then

$$(\lambda_{ijk})\notin M\left({}_3(c^{I(F)})(p),\ {}_3(c^{I(F)})(p)\right).$$

Proof. The result follows from Lemma 2.2. ■

Proposition 4.11 If $\Omega\Lambda^{-1}=(w_{ijk}\lambda_{ijk}{}^{-1})\in {}_3\ell_\infty^I(p)$ then $Z(\Lambda)\subset Z(\Omega)$ and the inclusion is proper, where $Z={}_3(c^{I(F)})(p),\ {}_3(c^{I(F)})(p).$ Proof. The inclusion $Z(\Lambda)\subset Z(\Omega)$ is obvious. To prove that the inclusion is proper, we cite a counter example.

Example 3.5. Let $A\in I_3,$ $p_{ijk}=\begin{cases}\frac{1}{3}, & ij\ \ (i,j,k)\in A\\ 3, & otherwise\end{cases}$

Define the sequence $\langle X_{ijk}\rangle$ as

For all $(i,j,k)\notin A,$

$$X_{ijk}(t)=\begin{cases}1+\dfrac{t}{(i+j+k)}, & for\ \ -(i+j+k)\le t\le 0\\ 1-\dfrac{t}{(i+j+k)}, & for\ \ 0<t\le (i+j+k)\\ 0, & otherwise\end{cases}$$

otherwise $X_{ijk}=\bar{0}.$

Taking $w_{ijk}=\dfrac{1}{(i+j+k)},\ \lambda_{ijk}=1$ for all $i,j,k\in N,$ we have $\langle X_{ijk}\rangle\in Z(\Omega)$ and $(w_{ijk}\lambda_{ijk}^{-1})$ is bounded. But $\langle X_{ijk}\rangle\notin Z(\Lambda)$ for $Z={}_3(c^{I(F)})(p),\ {}_3(c^{I(F)})(p).$ ■

Conclusion

For the development of any sequence space, convergence of that sequence space plays an important role. Ideal convergence is a generalization of the usual notation of convergence. In this article, we have introduced and studied some multiplier fuzzy real-valued ideal convergent triple sequence spaces. We have discussed some basic algebraic and topological properties of the introduced spaces. We hope that the results introduced in this article can be applied for further investigations from different aspects.

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